MODULAR ARCHITECTURAL GROUPINGS FROM ESCHER PERIODIC TESSELLATIONS

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ABSTRACT
One of the more interesting design techniques developed by Dutch graphic artist M.C. Escher consists in covering the plane with tiles containing patterns that repeats periodically. Modularity within shape grouping is extensively used, expressed by natural figures from the living world, and also from worlds of fantasy. This paper attempts to use Escher’s ideas as a source of inspiration to obtain modular shapes to conform groups with architectural issues.

The task is to satisfy design requirements and to get repeatable unitary shapes, whose geometric description is modularly manipulated within area as well as perimeter.

This should be done by two procedures:
1. from the components to the whole (from the tiles to the tiling): once the designer has defined a modular constructive unit (solving a particular situation), it is possible to repeat the unit to generate modular groups, knowing that they will fit perfectly among them, without gaps nor overlaps.
2. from the whole to the components (from the tiling to the tiles): defining a tessellation with the particular rules that drives close to the architectural solution, and getting the necessary units from the tiling.

There are multiple architectural themes on which this should be performed. School class-rooms, habitation buildings, shopping center sites, hotel rooms, are examples of this statement.
Analyzing procedures followed by the artist, particularly those using figures that tessellate the plane periodically, we’ll be able to generate tiles with architectural shape by the same way, applying different symmetry rules. Once the rules to generate shapes of tiles are known, we work within area and perimeter to satisfy modularity requirements and to convert the tiling as a geometric precise support for the insertion of architectural objects that follow predetermined dimensional patterns. In order to illustrate these ideas an example of grouping repeatable habitation units is presented.
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Introduction
A tiling or plane tessellation is any arrangement of polygons fitting together covering the whole plane just once, without overlapping neither letting empty spaces (interstices). Every side of each polygon belongs also to one other neighboring polygon.

If we look for the meaning of the word tessellation in the specialized literature (1),(2), we shall find a few synonymous such as mosaics, tiling, grids, reticulations, lattices, even space-filling surface patterns. In the text of this work we will use the words tessellation and tiling indistinctly.

The art of tessellating must have originated very early in the history of civilization and, throughout different periods, its techniques always attracted artisans and all kind of people who wish to apply geometric ideas in their work. Architects, decorators, painters, quilters, between others, have made use of tiles either as some imaginary support of shapes, as well as constructive units to be located on flat surfaces and the like.

Every known society seem to have emphasized different aspects while using tiling:
• Mediterranean cultures such as the Romans were concerned with portraying human beings and natural scenes in intricate mosaics, usually made up by very little pieces.
• Many floors and walls of stately Greek and Roman villas were tiled with the same patterns we know today as our most traditional patchwork designs, including triangles, quadrilaterals and hexagons.
• The Moors and Arabs characterized themselves by the use of a very few shapes and colors to build up complex geometric designs. Famous examples are to be seen in The Alhambra at Granada in Spain.
• Nowadays, by contrast with another cultures, square and rectangular tiles seem to be almost universal, because of its practicality and constructive easiness.
• M.C. Escher, (1898-1972), was a truly pioneer in the use of tessellation with a great sense of generalization, and this is the fact we must rescue and emphasize. He was an unusual artist, driven by a desire to solve problems which may seem more relevant to the mathematician than the printmaker (3). Working with symmetry operations (translations, rotations, glide reflections and their combination), he invented a method to cover the plane with organic living shapes, sometimes from real world, others from fantasy worlds. Motifs which frequently appear include fishes, lizards, birds, sea and landscapes. And here is where we ask this question : Why not? Why these (in some way architecturally strange forms composed by bizarre shapes) could not be a geometric support for an architectural application? Why could not these shapes be modular ones?. We should explore this modular design strategy employing periodic tessellation inspired by Escher’s art.
Concepts and definitions

Actually, for our cultural background, we can say the word “tessellation” comes from the Latin “tessellae”, which was the name given by the Romans to the small tiles used for pavements and walls in ancient Rome. A tessellae is precisely a tile that follows the restriction to fit with others without overlapping neither letting empty patches. Geometrically, a tile is a plane region with laminar structure, whose boundary is limited by a closed curve made up by line segments.

A regular tessellation is composed repeating the same shape again and again covering theoretically the whole Euclidean plane. If this is possible by only one kind of tile, the tiling is called monohedral. The word monohedral means that every tile in the tiling is congruent with another of the same tiling by means of a translation, rotation or reflection. More simply said, all the tiles are the same size and shape. In this way, if the tiling is monohedral, a prototype tile is available. We should call it a prototile. Also in the case that a set of tiles is congruent to another set of tiles of the same tiling, preserving the property of being a new composed tile covering the plane without gaps nor overlaps, we say that this set is a composed prototile.

![Figure 1: Simple prototiles.](image1)

![Figure 2: Composed prototiles: tilings (b),(c) and (d) are made up combining simple tiles from (a).](image2)

It is also possible to tessellate the plane with more than one kind of tile, and depending on the number of them, it will be called dihedral, trihedral, 4-hedral..., k-hedral tilings in which there are two, three, four, ...., k distinct prototiles.
When a tiling admits any symmetry operation besides the identity symmetry, then it will be called **symmetric**. If its symmetry group contains at least two translations in non parallel directions the tiling will be called **periodic**. In other words, a periodic tiling is one that admits a periodic repetition of its shape along two non parallel translations represented by a pair of vectors \(a, b\). In fact, the symmetry group contains all the translations \(na + mb\) where \(n\) and \(m\) are integers. The periodicity arises when shapes repeat themselves in regular distances and it is also called **cyclic distribution** of the plane surface. Its structure is denominated **lattice**. Starting from any fixed point \(O\), the set of images of \(O\) under the set of translations \(na + mb\) forms a lattice.

One of the most well known examples of a lattice is the set of points with integer coordinates belonging to the Euclidean plane, configuring a regular tiling formed by squares called the **unit square lattice**. Generalizing even more, a lattice can be regarded as consisting of the vertices of a parallelogram tiling. Thus, with every periodic tiling is associated a lattice, and the points of the lattice can be regarded as the vertices of a parallelogram tiling. This should be presented in many various manners. If we know the interior configuration of one parallelogram, it is possible to build the rest of the tiling by repeating this configuration in the whole set of parallelograms.
A lattice of points in the plane and some parallelograms whose corners coincide with points from the lattice. Each of these parallelograms is a prototile of a parallelogram tiling whose vertices map the original lattice. All such parallelograms have equal area. Any of them could be chosen as a prototile regarding that no lattice point lies in the interior or on the boundary of any parallelogram.

A periodic monohedral tiling containing a design pattern (a motif) that repeats in every tile interior.

One possible periodic parallelogram from the tiling indicated in (b).

There are a few more considerations to do about classifying tilings and tiles, in relation with symmetry and congruence (4). This will help to recognize which kind of tessellation we are dealing with. Let’s consider two tiles $t_1$ and $t_2$ of a tiling $T$. They are said to be equivalent if the symmetry group $S(T)$ contains a transformation that maps $t_1$ onto $t_2$. The set of all tiles of $T$ that are equivalent to $t_1$ is called the transitivity class of $t_1$. If all tiles of $T$ form one transitivity class, we say that $T$ is isohedral or tile-transitive. Let’s see a few examples of isohedral tessellations.

Although the distinction between isohedral tilings and monohedral tilings may seem slight, it is not. Finding and classifying all monohedral tilings is an unsolved problem, (even in the case tiles are convex polygons). Nevertheless, it is possible to describe and classify all the isohedral tilings, fact that marks the difference. The name a tiling will receive depends on the amount of transitivity classes it has. If $T$ is a tiling with precisely $k$ transitivity classes then $T$ is called $k$-isohedral.
If the symmetry group $S(T)$ of the tiling $T$ contains operations that map every vertex of $T$ onto any other vertex, then we say that the vertices form one transitivity class, or that the tiling is **isogonal**. Examples of this kind are the three regular tilings, those composed only by squares, only by equilateral triangles or only by hexagons. Also there are a lot of isogonal tilings that are not monohedral. See some examples in the following figure.

![Figure 7: Examples of isogonal tilings.](image)

A tiling is **$k$-isogonal** if its vertices form $k$ transitivity classes, where $k \geq 1$ is any integer.

A **monogonal** tiling is one which every vertex, together with its incident edges, forms a figure congruent to that of any other vertex and its incident edges. The distinction between isogonal and monogonal tilings is analogous to that between isohedral and monohedral tilings. Figure 7(a) is a dihedric tiling that also is monogonal. It is important to mention a last concept about classifying tilings: those in which every edge can be mapped onto any other edge by a symmetry of the tiling, are called **isotoxal**.

During his entire life Escher used this distribution system as the most important theme of his work, including in the geometrically pure tiles designs expressed by natural, organic shapes from the living world and also from worlds of fantasy and imagination. Such tiles can adopt any polygonal shape, segmented enough number of times to simulate a very smooth curve if necessary. This little trick allow us to obtain configurations that seem to be any kind of imaginable natural or artificial object, geometrically complex.
There are a lot of architectural themes in which once the designer has defined a building unit that solves a typical situation, it is possible to repeat the unit to conform groupings (this is an inductive-synthetic method of design process, which goes from the pieces to the whole).

In the same way, we can start from a group of shapes and recognize a single unit, or recognize a smaller group that solves a particular situation (this is a deductive-analytic method of design process, which goes from the whole to the components).

Classrooms in a school, dwelling in habitation sets, stores in a shopping center, hotel and motel rooms, are examples of this statement. If we can get a unitary repeatable shape, whose geometric description is modularly driven on its area and on its perimeter as well, satisfying design requirements, the digital work proposed in this article is effectively possible. The first step configuring architectural layouts is to determine a shape according to a mental prefiguration from who is conceiving it, exploding some main “force idea” or designing intention. On the other hand, almost every shape (two dimensional form) can be synthesized geometrically by means of some kind of polygon, generally non regular, particularly closed. This work proposes to obtain shapes of recognizable elements, natural or artificial, that preserving its geometrical structure (without altering its form), have a determined area and a modular perimeter. In other words, its perimeter has to be partitioned into modular line segments, without changing the original required area.

To obtain these shapes is necessary to work on the perimeter of basic polygons, the ones we are sure that tessellate the plane monohedrically. The polygons act like directress figures, and should be modified following very particular generating rules that let us get the tiles without gaps nor overlaps, once modified (5).

**Tiling with basic polygons**

Before we enumerate the generating rules to get tiles and tiling shapes Escher alike, it is convenient to analyze which kinds of basic polygons have the property to tessellate the plane monohedrically. Not only regular polygons tile
the plane, some non-regular ones can tessellate under a few restrictions. Besides, it is not strictly necessary that the directress figures were convex to tessellate the plane.

An obvious condition to tessellate with any figure is that arranging them around a vertex, the sum of the angles must fill 360 degrees. The following figures show two ways by which any triangle (even scalene) tessellate the plane.

Figure 9: Tiling generated by scalene triangles by the technique of rotations around midpoints.

Figure 10: Variation of the technique: rotating the triangle three times around the midpoints, then reflecting these triangles across the axis, creates a pattern that will tessellate.

The other cases by triangles are inside the general case. An isosceles with two equal sides and one not equal, also gives a parallelogram when we operate in the same manner. An equilateral triangle and therefore equiangular is a particular case of isosceles. A rectangular triangle working this way results in a tessellation of rectangular quadrilaterals.

Analyzing which quadrilaterals tessellate the plane, the conclusion is that every convex polygon with four sides do tessellate: squares, rhombs, rhomboids, parallelograms, rectangles, trapezes, trapezoids, scalene quadrilaterals. Some of them tile the plane by pure translation (square, rectangle, rhomb, parallelogram), the others require rotation or reflection besides translation. It is also possible to tile the plane with non-convex quadrilaterals, in the same manner triangles do.

Rotating a quadrilateral around the midpoints of its sides produces an arrangement that will tessellate. Once we have a pattern formed by four quadrilaterals, we can get the tiling by pure translation.

The sum of the interior angles of any pentagon is 540 degrees. Thus, it is not possible to fit all the angles of a pentagon into 360 degrees. Therefore, the technique of rotating about the midpoints of the sides will
not work with pentagons. Nevertheless, we can try to limit the technique with some geometric arguments that let pentagons tessellate the plane surface.

1. if two adjacent angles of a pentagon total 180 degrees, then the pentagon will tessellate in one of two ways: by translations or by reflections.
2. a special type of pentagon that has two non adjacent right angles (congruent 90 degrees angles) and two pair of sides of equal lengths (two pairs of congruent sides). Taking advantage of the equal side lengths and the convenient 90 degrees angles, it is possible to create a pattern where the right angles all meet at a single vertex.
3. if a side of a triangle is broken up in a way that the pentagon thus formed can join to itself, broken end to broken end, the pair of pentagons then acts like a parallelogram.

![Figure 11: Examples of tilings with non-regular pentagons.](image)

There are a lot of examples in nature that show tilings with regular hexagons (bees panel, trees cortex, chemical structures, etc). Also tessellate the plane those hexagons that have parallel sides two on two and, curiously, are the ones that dissecting them in two portions with a partition line crossing over the gravity center, let us get two quadrilaterals (we have already seen that they have the property to tile the plane). A pair of techniques can be applied to get non-regular hexagons that tile the plane surface. If we start with a parallelogram, and then create two sides from each of two opposite sides, we get some kind of non-regular hexagon. Making some specific movements it will tessellate by translations and by rotations.

Another way to make a tiling with hexagons consists in drawing first any quadrilateral, then divide one of the sides in halves and create new two sides from one of the halves. Rotating the new sides around the midpoint to replace the other half and removing the old half, we are done creating the hexagon. It is easy to see that this entire new shape can be rotated around the midpoint and we get two hexagons joined as a parallelogram, figure that can tessellate, using translation to repeat the pattern.

Some other regular polygons such as octagons and dodecagons have the property to tile the plane surface in combination with other regular polygons, but they cannot tile monohedrically. It is also possible to tile the plane with non regular polygons with more than six sides, and it is not difficult to obtain these shapes modifying basic figures.
Rules to generate “Escher tiles” from basic polygons.

As we have seen, choosing adequate directress basic polygons ensures the plane surface tiled periodically, by organic and other imaginary shapes. From this concept, we can announce the following generating rules which let us produce modifications on the basic polygons borders. The shapes obtained must admit design patterns in their interior, in the same way Dutch artist M.C. ESCHER used to.

1. **By translation.**
   Every portion cut on a side (concavity) is translated to the opposite parallel side (convexity). This rule can be applied to parallelograms and to hexagons with opposite parallel sides. The translation could be made only in the X direction, only in the Y direction, or both.

![Figure 12](image)

2. **By rotation with respect to the middle point of a side.**
   Every portion cut from the middle point of a side to one extreme (concavity) is added on the same side by a 180 degrees rotation, with center at the middle point of that side (convexity). This rule is applied to triangles and quadrilaterals.

![Figure 13](image)

3. **By rotation with respect to a vertex (hexagons and parallelograms)**
   Every portion cut on a side (concavity) of polygonal figures that have internal angles of 60 and 120 degrees, we turn it the mentioned angle and add this cut on the other side (convexity), with rotation center at this vertex. This rule is applied to hexagons and parallelograms. There is a restriction: in regular
figures such as a regular hexagon, the vertices center of rotation cannot be consecutive. It also should be applied to rhombs and rhomboids whose interior angles can have any value, with the same restriction: vertices center of rotation must be alternated.

4. **By rotation with respect to a vertex** (polygonal figures with one or more 90 degrees interior angle)
   Every portion cut on a side (concavity) of polygonal figures with an interior angle of 90 degrees, we turn it the mentioned angle and add this cut on the other side (convexity), with rotation center at this vertex. In case there is more than one 90 degrees angle the restriction is the same as before. This rule is applied to figures such as rectangular triangles, quadrilaterals and even pentagons with an angle of 90 degrees.

5. **By glide reflection with respect to a straight line.**
   This means that once a design has been defined on the side of a basic figure, it is translated along a predetermined distance, then it is reflected taking the mentioned straight line as the axe of reflection. This rule is applied to figures such as rectangles, squares, rhoms and isosceles triangles.

6. **By combination of translation plus rotation.**
   This rule is applied to rectangles, modifying a side and translating the modification to the opposite side. Then, every portion cut from the middle point of the perpendicular side to one extreme (concavity) is added on the same side by a 180 degrees rotation, with center at the middle point of that side (convexity).
7. **By combination of translation plus glide reflection.**
This rule is applied to rectangles, modifying a side and translating the modification to the opposite side. Then, a glide reflection is applied to the other two sides: defined a design it is translated along a predetermined distance (the length of the rectangle), and then it is reflected taking a central straight line as the axe of reflection.

8. **By combination of rotation plus glide reflection**
Every portion cut from the middle point of a side to one extreme (concavity) is added on the same side by a 180 degrees rotation, with center at the middle point of that side (convexity). Afterwards a reflection axe is defined between two sides where the rotation wasn’t applied, and a glide reflection is executed. This rule is applied to isosceles triangles and quadrilaterals.

**Architectural Application**

From an architectural point of view, once the shape has been approximately preconceived, the procedure could be divided in two stages:
1.- the obtention of the functional required area
2.- the partition of the perimeter into modular line segments

The first stage is about locating the coordinates of the baricenter (gravity center of the figure), take them as the basis point to change shape dimensions, and by a proportionality relationship between original area and required area, we must find the coordinates of all the new points belonging to the perimeter, of the new scaled shape.

The second stage is about polygons resolution, working their perimeter in a modular fashion. Every polygon is defined by means of the coordinates of its vertices and the direction of the edges that join each pair of vertices. From each perimetral edge it is possible to obtain a partition with a great amount of new segments, each of them with a linear modularity assigned. Although a few vertices of these new segments will modify their location once we made the modularity correction, the vertices from the original polygon must keep their coordinates.

The procedure starts assigning a linear value to the module. Then the distance between two vertices of the polygon is “forced” to take a multiple value of the assigned module, respecting the required shape. It could be a straight line segment or a circular curve arc, on each perimetral portion. The biggest difficulty shows up when we get to the last portion of each polygon or polygonal section, when it is required that even if we advance from left to right as well as from right to left, two points must coincide without altering the modularity. This should be solved by the intersection of two circumferences which centers are the preceding point and the proceeding point to the one where it is required to obtain modular coincidence.

Figure 17: (a) Tiling Escher alike where the tiles simulate horses heads. Perimeter and area are modulated for the dwelling units. (b) A dwelling unit on the indicated configuration. The black dots are disposed at the same distance on the perimeter: equal modules letting insert modular architectural objects.

Working this way on every side of the basic polygon, with each segment of the modified polygonal, we obtain a 2D shape that encloses a predetermined area (required by the designing program) and also a linear modulation. Any
architectural element we want to insert to configure an electronic model, will have a linear modularity congruent with the generated tiles. To give an example, a grouping of little dwelling units is presented, satisfying a predetermined area and modularity.

Figure 18: A grouping of four dwelling units.

Being Design a subject that necessarily deals with shapes and forms, the use of geometry in a rational way, preserving economy, is worthwhile. An alternative consists in proposing procedures that help to obtain modularity within area and perimeter, letting the designer manipulate forms, without altering programmatic requirements. Operating with digital algorithms (6),(7) that take tessellations of the plane as a geometric basis, it is possible to redesign tiles and tilings Escher alike, to look for modularity and functional efficiency, with a great deal of design freedom.

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